Comment on "Dynamics of wetting fronts in porous media"

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A new method to model unsaturated flow in porous media was presented in Phys. Rev. E **58**, R5245 (1998). We analyze the proposed approach and illustrate some significant shortcomings.

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Mitkov, Tartakovsky, and Winter [1], (MTW) have undertaken the noble task of trying to develop an improved description of the dynamics of wetting front propagation in porous media. In particular, they propose a phenomenological approach that purports to take advantage of the dynamic nature of the capillary pressure vs saturation relation. They claim that because their equations provide some level of agreement with standard approaches, their proposed model is valid and worthy of further development and experimental support. We disagree with this proposition and will point out significant shortcomings of the suggested equations.

The equations proposed by MTW are intended to describe the advancement of a wetting front in a porous medium. The standard model for unsaturated flow, which includes as a subset of the general model a description of the rate of movement of a wetting front given appropriate specification of auxiliary conditions, is Richards' equation (RE) [2]. MTW discuss RE and compare results from their proposed approach to this model.

In the notation of MTW, the standard conservation of mass equation for wetting-phase flow in a porous medium in the absence of internal sources or sinks or interphase mass transfer is

$$\frac{\partial(\rho\omega\theta)}{\partial t} = -\nabla\cdot(\rho\mathbf{q}), \tag{1}$$

where ρ is the density of the wetting phase; ω is the porosity of the porous medium; θ is the saturation of the wetting phase, where $\theta\in[0,1]$; and $\mathbf{q}$ is the wetting fluid flux vector or the volume of wetting fluid flowing per bulk area per time. For the special case of a constant density wetting phase, Eq. (1) simplifies to

$$\frac{\partial(\omega\theta)}{\partial t} = -\nabla\cdot\mathbf{q}. \tag{2}$$

MTW define $\mathbf{q}$ in the usual way for RE as

$$\mathbf{q} = -K(\theta)\nabla(\psi - x_3), \tag{3}$$

where K is the saturation-dependent conductivity of the medium, x_3 is the vertical coordinate, and ψ is the pressure head of the water phase. Substitution of Eq. (3) into Eq. (2) yields the standard mixed form of RE, which is

$$\frac{\partial(\omega\theta)}{\partial t} = \nabla\cdot[K(\theta)\nabla(\psi - x_3)]. \tag{4}$$

In contrast, MTW state RE as

$$\frac{\partial\theta}{\partial t} = \nabla\cdot[K(\theta)\nabla(\psi - x_3)]. \tag{5}$$

Richards developed Eq. (4) by combining Darcy's law, an approximate momentum equation for slow flow, with the conservation of mass equation. RE conserves mass. Since Eq. (5) is not consistent with Eq. (4), it does not conserve mass and is in error.

Although RE is based on conservation principles, it has some shortcomings in application. These difficulties stem from assumptions made in developing the equation: (i) the physics of the air phase are not explicitly treated; (ii) the equilibrium relationships among ψ, θ, and K are hysteretic at equilibrium, depending upon wetting and drying history; and (iii) the dynamics of these relationships are usually ignored. For these and other reasons, it is worthwhile to search for a more general formulation that may describe better the physics of unsaturated flow.

Apparently with this thought in mind, MTW proposed the pair of equations

$$\frac{\partial\psi}{\partial t} = D\nabla^2\psi - \frac{1}{S}\frac{\partial\theta}{\partial t} - D\frac{\partial\theta}{\partial x_3}, \tag{6}$$

$$\tau\frac{\partial\theta}{\partial t} = W^2\nabla^2\theta + [2\theta - 1 - \lambda(\psi - \psi_f)\theta(1-\theta)]\theta(1-\theta), \tag{7}$$

where $S = \rho g\omega(\beta_f - \beta_s + \beta_p)$, ρ is density of the wetting phase, g is the gravitational acceleration constant, β_f is the compressibility of the fluid, β_s is the compressibility of the solid grains, β_p is the compressibility of the pores, $D = K_s/S$, and K_s is the conductivity of the fully saturated medium. These equations are not derived but are said to be "phenomenological in the sense that presently we do not provide a rigorous physical motivation for the nonlinear source term on the right-hand side of Eq. (3)," which corresponds to Eq. (7) above.

Although phenomenological equations may be useful in some instances to describe observed system behavior, they must not violate physical principles such as the fundamental conservation laws. As a first test of a proposed system of equations, their consistency with physical principles should be examined. The authors do consider Eq. (6) for the case of an incompressible fluid and medium and claim that it conserves mass for this case. However this assertion requires additional consideration. For the incompressible case, MTW indicate that Eq. (6) reduces to

$$\frac{\partial\theta}{\partial t}=K_s\left(\nabla^2\psi-\frac{\partial\theta}{\partial x_3}\right) \tag{8}$$

or, when K_s is constant,

$$\frac{\partial\theta}{\partial t}=\boldsymbol{\nabla}\cdot[K_s(\nabla\psi-\theta\mathbf{e}_3)], \tag{9}$$

where $\mathbf{e}_3$ is a unit vector in the x_3 direction. Although MTW claim this to be a mass-conserving equation, it is not consistent with either the RE, Eq. (4), which is known to be mass conserving, or MTW's equation, Eq. (5). Therefore, Eq. (8) violates conservation of mass. Further, the steady state form of Eq. (8) implies a relation between ψ and θ that is independent of medium and fluids. This is not consistent with observed behavior of multiphase flow in porous media.

Equation (7) should also be examined for its agreement with conservation principles. Such agreement should be met in addition to the stated objective of MTW of providing an approach to describe wetting front propagation. This equation is a diffusion equation with a reaction term. This is a surprising equation to use in describing water content; it implies that water may appear or disappear through some kind of generation process. Phase change of the wetting phase, however, is not considered in the physical statement of the problem considered by MTW. It is interesting to note that the proposed phase field model has been applied in the field of solidification, where phase change is an essential part of the physics. Additionally, if one considers Eq. (7) for a system without spatial or temporal gradients in θ, it is easy to see that the only solutions allowed for this static system are $\theta=0$, $\theta=1$, or $\psi=(2\theta-1)/[\lambda\theta(1-\theta)]+\psi_f$. Certainly this is overly restrictive as other static situations are commonly found. Additionally, Eq. (7) requires a violation of the principle of mass conservation when the last term is nonzero. It thus presents a description of system behavior that is not consistent with observations. Experimental study would thus seem unwarranted.

Neither Eq. (6) [or more precisely its special case of Eq. (8)] nor Eq. (7) is physically or phenomenologically correct. If progress is to be made in modeling unsaturated flow, the work must not violate conservation principles. A general approach in which coefficients appearing in conservation equations have dynamic functional dependence may lead to improved parametrizations. Thus, although an alternative approach to RE is warranted, the work of MTW has serious theoretical shortcomings and seems incapable of contributing to an improved understanding of the physics of unsaturated flow in porous media.

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[1] I. Mitkov, D. M. Tartakovsky, and C. L. Winter, Phys. Rev. E **58**, R5245 (1998).

[2] L. A. Richards, Physics (N.Y.) **1**, 318 (1931).